Optimal Seismic Reflectivity Inversion: Data-Driven \( \ell_p \)-Loss-\( \ell_q \)-Regularization Sparse Regression

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Abstract—Seismic reflectivity inversion is widely applied to improve the seismic resolution to obtain detailed underground understandings. Based on the convolution model, seismic inversion removes the wavelet effect by solving an optimization problem. Taking advantage of the sparsity property, the \( \ell_1 \)-norm is commonly adopted in the regularization terms to overcome the noise/interference vulnerability observed in the \( \ell_p \)-loss minimization. However, no one has provided a deterministic conclusion that \( \ell_1 \)-norm regularization is the best choice for seismic reflectivity inversion. Instead of using an unproved fixed regularization norm, we propose an optimal seismic reflectivity inversion approach. Our method adaptively adopts an \( \ell_p \)-loss-\( \ell_q \)-regularization (i.e., \( \ell_{p,q} \)-regularization) for \( p = 2, 0 < q < 1 \) to estimate a more accurate and detailed reflectivity profile. In addition, we employ a \( K \)-fold cross-validation (CV)-based approach to obtain the optimal damping factor \( \lambda \) to further improve the seismic inversion results. The letter starts with the introduction of nonconvex constraint for seismic inversion and the necessity of the \( \ell_q \)-norm regularization. Then, the majorization-minimization and CV algorithms are briefly described. The performance of the proposed seismic inversion approach is evaluated through synthetic examples and a field example from the Bohai Bay Basin, China.

Index Terms—cross-validation (CV), damping factor, \( \ell_{p,q} \)-regularization, optimal regularization, seismic reflectivity inversion.

I. INTRODUCTION

RESOLUTION is an unavoidable topic in seismic interpretation, which is the ultimate goal for all seismic processing steps. High seismic resolution helps to characterize the correlation between the geological structures and geophysical images, which is important for the reservoir deposition analysis, especially thin-bed reservoirs [1].

Based on the seismic convolution model, Chen and Wang [2] improved the seismic resolution via a wavelet scaling method. Likewise, Zhou et al. [3] extended the seismic spectrum using a nonstationary wavelet estimation. A seismic reflectivity obtained from seismic amplitude inversion has been used in the applications with little/weak well controls [4]. For regularized inversion, besides the \( \ell_2 \)-norm constraint, \( \ell_1 \)-norm regularization is more popular in sparse spike inversion [5], [6]. Theoretically, \( \ell_0 \)-norm regularization should also be a good sparsity measurement, but the actual seismic signals do not always show 0 values, so finding exact \( p \) nonzero elements in signal, i.e., \( \| x \|_0 = p \) is not a computationally feasible and proper constraint in the real world [7]. Signal sparsity properties show different highlights [8], so a data-driven sparsity measurement is more suitable for the seismic reflectivity inversion problem [9], [10]. The convergence of \( \ell_q \) \((0 < q < 1)\)-norm regularization has been proven by [11] and its advantages were also demonstrated. There are some well-established algorithms to solve the nonconvex or concave optimization problem related to \( \ell_q \)-regularized inversion, for examples, cyclic descent algorithm [12], reweighted \( \ell_1 \) minimization [13], iteratively reweighted algorithms [14], and so on.

However, to our best knowledge, the sparsity regularization selection for seismic reflectivity inversion has not been generally discussed. Although there are discussions with the damping factor \( \lambda \) selection, such as [10], they are only related to either \( \ell_1 \)- or \( \ell_2 \)-norm. In the previous work [15], a \( \ell_q \)-norm (\( 0 < q < 1 \)) regularized optimization has been applied to the seismic reflectivity inversion. Compared to [15], this letter proposes a general data-driven seismic reflectivity inversion approach. We extend \( \ell_q \)-norm (\( 0 < q < 1 \)) to a \( \ell_p \)-loss-\( \ell_q \)-regularization (\( \ell_{p,q} \) regularization) framework, which allows us to consider a wide range of loss functions. The possible choices of loss functions include the \( \ell_p \)-losses and prediction losses [11]. In this letter, we study the \( \ell_2 \)-loss as an illustration. Here, we also explicitly define the way to obtain the optimal \( q \), where the majorize-minimization (MM) algorithm is described. In addition, \( K \)-fold cross-validation (CV) is adopted for generality. We establish an adaptive seismic inversion approach including the optimal regularization term selection as well as the adaptive damping factor determination. We introduce the theories and algorithms first. Then, the effectiveness of the proposed method is proven through synthetic examples and a field application from Bohai Bay Basin, China.

II. THEORY

A. Seismic Convolution Model

After removing the undesirable interferences, seismic data can be modeled as the convolution between the source wavelet...
\( w(t) \) and reflectivity series \( r(t) \) in the form as [10], [15], [16]

\[
s(t) = \int_{-\infty}^{\infty} w(t) r(t - \tau) d\tau + \epsilon(t)
\]

where \( \epsilon(t) \) is an additive random noise. Using a convolution symbol, we can rewrite it as

\[
s(t) = w(t) * r(t) + \epsilon(t).
\]

Then, a discrete matrix format realized at samples \( t = 1, 2, \ldots, M \) can also be formulated [17]

\[
s = Wr + \epsilon
\]

where \( s = [s_1, \ldots, s_M]^T \) is the observation, \( W \) is the wavelet kernel matrix which can be estimated from the seismic power spectrum [16], and \( r = [r_1, \ldots, r_M]^T \) is the reflectivity series.

**B. Seismic Inversion**

The seismic convolution model forms a linear relationship between the recorded seismic data and reflectivity series. The solution of (3) is nonunique, which necessitates certain constraint to achieve stable. In general, the objective function can be formulated as

\[
\phi(q, \lambda) = ||s - Wr||_p^p + \lambda ||r||_q
\]

where \( ||s - Wr||_p^p = \sum_{i=1}^{M} |s(t) - w(t) * r(t)|_p^p \) and \( ||r||_q = (\sum_{i=1}^{M} |r|^q)^{1/q} \) (\( q \) indicates the constraint condition), and \( \lambda \) is the damping factor [12]. The common choice for the loss function is the \( \ell_2 \)-loss, i.e., \( p = 2 \).

**C. \( \ell_q \)-Norm Regularization**

Seismic inversion usually adopts either \( \ell_1 \)- or \( \ell_2 \)-norm regularization. This nonconvex constraint is based on the common assumption that the reflectivity series \( r(t) \) is sparse. However, accurately identifying the sparsity of the signal is not easy.

The \( \ell_0 \)-norm, which measures the number of nonzero of \( r: ||r||_0 = \{r_j \neq 0, j = 1, \ldots, N\} \), is not put into consideration. It asks to find the “best subset” among all possible sparsity candidates that make it impractical for seismic reflectivity due to intractable computational cost and difficulty in choosing number of nonzero elements. To loosen the constraint, the \( \ell_1 \)-norm regularization, i.e., Lasso regression [18], is easily calculated with sparsity constraint. Frank and Friedman [19] built the connection between \( \ell_0 \)- and \( \ell_2 \)-regularizations. Because the convex \( \ell_q \)-regularization with \( q > 1 \) does not hold the sparsity property, which is necessary for seismic inversion, we need to pick an optimal \( q \) from (0, 1). As shown in Fig. 1, different regularization norms produce various constraint on sparsity.

In order to obtain the optimal choice of \( q \), for a given \( \lambda \), we can formulate the objective function as

\[
\hat{q} = \arg \min_{q \in Q} \phi(q, \lambda) \Rightarrow \hat{q} = \arg \min_{q \in Q} \phi(q)
\]

where \( Q \) is a predefined nonempty set containing possible \( q \) values. This problem can be solved through a coordinate-wise optimization method.

**D. Majorize-Minimization Algorithm**

We propose to use the MM algorithm [20] to solve the nonconvex optimization problem in equation (4) with \( 0 < q < 1 \). As an iterative optimization method, the MM algorithm consists of two steps in one iteration. In the first majorization step, a surrogate function \( g(q|q_k) \) is employed to locally approximate the objective function with their differences minimized at the current point. In other words, the surrogate upperbounds the objective function up to a constant \( c_k \). Then, in the minimization step, the surrogate function is minimized. The sequence \( \phi(q_k) \) is nonincreasing since \( \phi(q_{k+1}) \leq g(q_{k+1}|q_k) - c_k \leq g(q_k|q_k) - c_k = \phi(q_k) \). The procedure is shown in Algorithm 1. The computational complexity of the MM algorithm is \( O(M^3) \).

**E. Damping Factor Selection**

The damping factor \( \lambda \) in (4) controls the sparsity of the recovered signal. We propose to use the CV approach [21] to choose the optimal value of the damping factor, which works for parameter selection according to its statistical performance. CV provides the optimal bias-variance tradeoff according to its statistical performance. CV provides the optimal bias-variance tradeoff under the setting of (4). The \( K \)-fold CV is adopted roughly as follows. First, randomly split the data into \( K \) equal clusters. Second, the \( k \)th cluster (\( k = 1, 2, \ldots, K \)) is used for testing the optimization model fit using the data from other cluster. Third, the CV prediction error for each \( \lambda \) is calculated. Then, repeat the random selection. In the end, combine all the \( K \) prediction
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TABLE I
SIMILARITY BETWEEN THE GROUND TRUTH AND INVERTED REFLECTIVITY PROFILE FROM 100 TIMES MONTE CARLO EXPERIMENTS OF EVERY SITUATION. (“STD” STANDS FOR STANDARD DEVIATION)

<table>
<thead>
<tr>
<th>SNR (dB)</th>
<th>$\ell_2$</th>
<th>$\ell_1$</th>
<th>$\ell_{p,q}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>mean</td>
<td>std</td>
<td>mean</td>
</tr>
<tr>
<td>20</td>
<td>0.5038</td>
<td>0.0625</td>
<td>0.8602</td>
</tr>
<tr>
<td>10</td>
<td>0.4991</td>
<td>0.0726</td>
<td>0.8362</td>
</tr>
<tr>
<td>3</td>
<td>0.4701</td>
<td>0.0670</td>
<td>0.8081</td>
</tr>
<tr>
<td>0</td>
<td>0.4292</td>
<td>0.0650</td>
<td>0.7896</td>
</tr>
<tr>
<td>-3</td>
<td>0.3914</td>
<td>0.0694</td>
<td>0.7417</td>
</tr>
<tr>
<td>-10</td>
<td>0.3685</td>
<td>0.0690</td>
<td>0.5156</td>
</tr>
</tbody>
</table>

The errors together to obtain the CV estimate

$$CV(\lambda) = \frac{1}{K} \sum_{k=1}^{K} (s_k - W_k \hat{r}(-k)(\lambda))^2$$

leading to the chosen optimal $\hat{\lambda} = \arg\min_{\lambda \in [a, b]} CV(\lambda)$, where $[a, b], a, b \in \mathbb{R}_+$ is the given range of $\lambda$.

III. EXPERIMENTS

A. Synthetic Example

We first test the proposed method in numerical experiments. Compared with synthetic experiments in [10] and [15], we designed the sparse reflectivity series randomly instead of using fixed positions, which makes more sense for the nonstationary seismic applications.

Table I shows the comparisons of the similarity measurement between ground truth and inverted reflectivity with different SNRs. $\ell_2$, $\ell_1$, and $\ell_{p,q}$-regularizations are applied to invert the random reflectivity profile, respectively. SNR varies from 20 to $-10$ dB, which covers possible SNR situations of field applications, and Monte Carlo experiments are conducted 100 times for every different situation. Note that the proposed $\ell_{p,q}$ method achieves the highest inversion accuracies at different noise levels with relatively stable performances. Specifically, when the data have a low SNR, the proposed regularization constraint produces the most robust results.

To visualize the results, Fig. 2(a) shows one of synthetic Monte Carlo reflectivity models consisting of randomly located, sparse reflectivity series. Based on the convolution model, we obtain the seismic data by convolving the reflectivity profile with a 30-Hz Ricker wavelet. The synthetic seismic trace is shown in Fig. 2(b). Fig. 2(c) displays a noisy seismic trace with 10-dB random noise. Fig. 2(d)–(f) demonstrates the inverted results from the noisy seismic data based on the regularizations of $\ell_2$, $\ell_1$, and proposed $\ell_{q}$-norms. Our proposed $\ell_q$ method and $\ell_1$-norm successfully recovered the sparsity of the reflectivity, where $\ell_2$-norm failed to do so. Compared to $\ell_1$-norm, the proposed $\ell_q$ method captured more detailed information, such as small-amplitude picks, successive picks, than $\ell_1$-norm method. Due to the high nonconvexity of the $\ell_q$-regularization, our proposed method distinguishes the true nonzero reflectivity components from noises by projected observed data onto a $\ell_q$-ball. By searching the $\ell_q$-ball using the MM algorithm, the true reflectivity can be efficiently recovered from the noisy data. The results demonstrated that, under the noise contamination, the $\ell_q$ method outperforms the popular seismic inversion techniques such as $\ell_1$- or $\ell_2$-norm regularized inversion.

B. Field Application

We apply the proposed method to seismic field data from the Bohai Bay Basin, China. The target reservoir in this area is the siliciclastic reservoir, where the thin-bed structure develops, so the high-resolution interpretation, especially the reflectivity inversion, is of great importance.

We display the original seismic amplitude in Fig. 3. Fig. 4 shows the inverse result of $\ell_2$-norm regularization with fivefold CV. We see that it fails to remove the noise.
Fig. 4. Seismic inversion using $\ell_2$-norm.

Fig. 5. Seismic inversion using $\ell_1$-norm.

Fig. 6. Seismic inversion using $\ell_q$-norm.

Fig. 7. Seismic inversion using $\ell_q$-norm without optimal regularization parameter selection.

Fig. 8. Comparison among the reflectivities calculated from the sonic logs, the inverted reflectivity based on $\ell_1$- and $\ell_q$-norms. (The trace number is 209.)

from observations or recover the sparse reflectivity series. Compared to $\ell_2$ method, $\ell_1$ and $\ell_q$ methods shown in Figs. 5–7 provide more sparse recovered reflectivity series. The optimal $\ell_q$ inverse result with $q = 0.7$ is shown in Fig. 6. We see that our proposed $\ell_q$ method recovered more detailed information when comparing to the $\ell_1$ method, and more thin beds can be characterized now. Note that the damping factors for both methods are chosen using fivefold CV.

To demonstrate the effectiveness of choice of damping factors, we compare the optimal $\ell_q$ inversion (Fig. 6) with an over regularized $\ell_q$ inversion (Fig. 7). We see that a proper regularization, i.e., finding the optimal damping factor using CV, will help keeping the continuity of the inversion result while maintaining the sparsity.

To further evaluate the inverted results, we compare the $\ell_1$- and $\ell_q$-norms inversion reflectivity profiles with the reflectivity calculated from the sonic logs, shown in Fig. 8. The results from the proposed $\ell_q$-norm inversion show more details and have better correspondence with the reflectivity from well logs. In addition, we also calculate the relative similarity coefficient, $\rho(\cdot, \text{Refl}_{\text{log}})/\rho(\text{Seismic}, \text{Refl}_{\text{log}})$, between the inverted reflectivity and the sonic reflectivity with $\rho(\cdot)$ as the cross correlation function, which are 0.5843 with the $\ell_q$-norm and 0.4347 with the $\ell_1$-norm. Therefore, from both qualitative and quantitative comparisons, the proposed method is better than the existing methods.

**IV. CONCLUSION**

We propose a novel sparse reflectivity inversion method with $\ell_{p,q}$-norm regularization and optimal damping factor selection. The nonconvex constraints have drawn a lot of attention because it satisfies the mathematical assumptions of seismic reflectivity series. Illustrating the case of $p = 2$, we adaptively adopt the $\ell_q$-norm ($0 < q < 1$) constraint. In addition, because the estimates of $q$ and $\lambda$ value are fully data driven, we believe the inverted results are relatively optimal. The synthetic examples verify that the proposed method can eliminate the effects of wavelet effectively and
can obtain the stable reflectivity series, even with moderate random noises. Furthermore, the derived reflectivity in the field example shows high correspondence with the reflectivity calculated from well logs.

**REFERENCES**


